RA 1

TAN

1. Cadinality of Sets

Given two sets A and B, how to compare the number of elements in A and B?

Let's begin with finite sets. Given a finite set A, the **cardinality** of A is defined to be the number of elements in A and denoted by $\overline{\overline{A}}$. Now let A, B be two finite sets and $\overline{\overline{A}} = \overline{\overline{B}}$, then there is a bijection between A and B. This motivates us to give the definition of the cardinality of any sets.

Definition 1.1. *Let A, B be two sets. We say A and B have the same cardinality and write $A \sim B$ if there exists a bijection $f : A \to B$.*

One can check that $\sim$ is a equivalence relation, and thus we say A and B have the same cardinality and write $\overline{\overline{A}} = \overline{\overline{B}}$ if $A \sim B$.

Lemma 1.1. *Let X, Y be two sets and $f : X \to Y, g : Y \to X$ be two maps. Then we have decompositions $X = A \cup A', Y = B \cup B'$, where $f(A) = B, g(B') = A'$ and $A \cap A' = B \cap B' = \emptyset$.*

Proof. If $f(X) = Y$, then the conclusion is trivial. Next we assume that $f(X) \neq Y$. Let $\Gamma = \{E \subseteq X : E \cap g(Y - f(E)) = \emptyset\}$ and $A = \bigcup_{E\in\Gamma} E$. Now for any $E \in \Gamma$, we have $E \cap g(Y - f(A)) = \emptyset$ and thus $A \cap g(Y - f(A)) = \emptyset$. As a consequence, A is the maximal element in Γ.

Now let $B = f(A), B' = Y - B$ and $A' = g(B')$. Then $Y = B \cup B'$ and $A \cap A' = \emptyset$. If $A \cup A' \neq X$, then there exists $x_0 \in X$ and $x \notin A \cup A'$. Let $A_0 = A \cup \{x_0\}$, then we have $B = f(A) \subseteq f(A_0)$ and $Y - f(A_0) \subseteq B'$. Thus, $g(Y - f(A_0)) \subseteq A'$. Since $A' \cap A_0 = \emptyset$, $A_0 \cap g(Y - f(A_0)) = \emptyset$, implying $A_0 \in \Gamma$, which contradicts the fact that A is maximal. Thus, we have $A \cup A' = X$, completing the proof. □

Theorem 1.1 (Cantor-Bernstein)**.** *If there exists two injections $f : X \to Y$ and $g : Y \to X$, then $X \sim Y$.*

Proof. By the previous lemma, we have $X = A \cup A', Y = B \cup B', f(A) = B, g(B') = A'$. Then $f : A \to B$ and $g^{-1} : A' \to B'$ are bijections. Thus, we get a bijection

$$F(x) = \begin{cases} f(x) & , x \in A, \\ g^{-1}(x) & , x \in A'. \end{cases}$$

As a consequence, $X \sim Y$. □

Now suppose $\overline{\overline{A}} = \alpha, \overline{\overline{B}} = \beta$. If there exists an injection from A to B, then we write $\alpha \leq \beta$. If $\alpha \leq \beta$ and there exists no bijection between A and B, then we write $\alpha < \beta$. If $\alpha \leq \beta$ and $\beta \leq \alpha$, then by Cantor-Bernstein's Theorem, we have $\alpha = \beta$.

Let $\mathbb{N}$ be the set of natural number and $\aleph_0$ be its cardinality. A set is said to be countable if its cardinality is $\aleph_0$.

In class we have shown that the set of rational number $\mathbb{Q}$ is countable. Next we show that the set of real number $\mathbb{R}$ is not countable.

Theorem 1.2. *The set $(0,1]$ is not countable.*

Proof. For each number $x \in (0,1]$, consider the binary expansion $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$, and we restrict the sequence $\{a_n\}_{n=1}^{\infty}$ such that there infinite $a_n = 1$. Then we get a bijection between $(0,1]$ and binary decimals in which the digit 1 appears infinitely. Now we rewrite $x = \sum_{i=1}^{\infty} 2^{-n_i}$, where $\{n_i\}_{i=1}^{\infty}$ is a sequence of positive integral number such that $n_{i+1} > n_i$ for each $i \geq 1$. Now let $k_1 = n_1$ and $k_i = n_i - n_{i-1}, i \geq 2$, then $\{k_i\}_{i=1}^{\infty}$ is a sequence of natural number. Let $\mathcal{N}$ be the set of sequences of natural number, then we get a bijection between $(0,1]$ and $\mathcal{N}$.

Now suppose that $(0,1]$ is countable, then $\mathcal{N}$ is countable and we can write $\mathcal{N} = \{\vec{k}_i\}_{i=1}^{\infty}$, where $\vec{k}_i = (k_j^{(i)})_{j\in\mathbb{N}}$. However, it is impossible, since the element $\vec{k} = (k_j^{(j)} + 1)_{j\in\mathbb{N}} \in \mathcal{N}$ but $\vec{k} \neq \vec{k}_i$ for any $i \geq 1$. Thus, $\mathcal{N}$ is not countable, meaning that $(0,1]$ is not countable. □

Since $(0,1) \subseteq (0,1] \subseteq \mathbb{R}$, and one can construct a bijection between $(0,1)$ and $\mathbb{R}$ easily, we have that $\overline{\overline{\mathbb{R}}} = \overline{\overline{(0,1]}}$ by Cantor-Bernstein's Theorem and $\mathbb{R}$ is not countable. We denote the cardinality of $\mathbb{R}$ by c or $\aleph_1$. One can see that $\aleph_0 < \aleph_1$.

Theorem 1.3. *Let $A \neq \emptyset$ be a set, and $2^A = \{B : B \subseteq A\}$. Then $\overline{\overline{A}} \neq \overline{\overline{2^A}}$.*

Proof. Suppose that $A \sim 2^A$, then there exists a bijection $f : A \to 2^A$. Let $B = \{x \in A : x \notin f(x)\}$. Then there exists $y \in A$ such that $B = f(y) \in 2^A$. If $y \in B$, then by the defnition of B we have $y \notin f(y) = B$, which is impossible. However, if $y \notin B$, then still by the definition of B, $y \in f(y) = B$ and get a contradiction.

Thus, there exists no bijection between A and 2^A, meaning that $\overline{\overline{A}} \neq \overline{\overline{2^A}}$. □

Now define $g : A \to 2^A$ by $g(x) = \{x\}$ for all $x \in A$, then we get an injection mapping A into 2^A. Thus, we have $\overline{\overline{A}} < \overline{\overline{2^A}}$.

Let's end this section with a question. For an infinite set X, do we have $\overline{\overline{X}} = \overline{\overline{X \times X}}$?

2. Baire's Theorem

In class we have defined open and closed sets in $\mathbb{R}^n$ and obtained some basic properties of them. For instance, the union of open sets is open and the intersection of closed sets is closed. We'll extend the content in this section.

Definition 2.1. *If G is a countable intersection of open sets, meaning that $G = \cap_{n=1}^{\infty} G_n$, where G_n is open for all n, then we say G is a G_δ set. If $F = \cup_{n=1}^{\infty} F_n$ is a countable union of closed sets, then we say F is an F_σ set.*

Theorem 2.1 (Baire). *Let $E \subseteq \mathbb{R}^n$ be an F_σ set and write $E = \cup_{k=1}^{\infty} F_k$, where F_k is closed for all k. If for each k, F_k contains no interior point, so does E.*

Proof. Suppose that x_0 is an interior point of E, then there exists $\delta_0 > 0$ such that $\overline{B_{\delta_0}(x_0)} \subseteq E$. Since F_1 has no interior point, there exists $x_1 \in B_{\delta_0}(x_0)$ such that $x_1 \notin F_1$. Thus, one can choose $0 < \delta_1 < 1$ such that $\overline{B_{\delta_1}(x_1)} \cap F_1 = \emptyset$ because F_1 is closed. By choosing δ_1 small enough, we can assume $\overline{B_{\delta_1}(x_1)} \subseteq B_{\delta_0}(x_0)$. Similarly, we can get $\overline{B_{\delta_2}(x_2)} \cap F_2 = \emptyset$ and $\overline{B_{\delta_2}(x_2)} \subseteq B_{\delta_1}(x_1)$, where $0 < \delta_2 < \frac{1}{2}$. Repeat this process, we can get a sequence of point $\{x_k\}_{k=1}^{\infty}$ and a sequence of positive number $\{\delta_k\}_{k=1}^{\infty}$, such that $\overline{B_{\delta_k}(x_k)} \subseteq B_{\delta_{k-1}}(x_{k-1}), \overline{B_{\delta_k}(x_k)} \cap F_k = \emptyset$ and $0 < \delta_k < \frac{1}{k}$. Noting that $x_l \in B_{\delta_k}(x_k)$ when $l > k$, we have $|x_l - x_k| < \delta_k < \frac{1}{k}$. Thus, $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^n$. As a consequence, there exists $x \in \mathbb{R}^n$, such that $\lim_{k\to\infty} |x_k - x| = 0$. Now we have $|x - x_k| \leq |x - x_l| + |x_l - x_k| < |x - x_l| + \delta_k$ for $l > k$. Letting l tend to infinity, we have $|x - x_k| \leq \delta_k$ and $x \in \overline{B_{\delta_k}(x_k)}$. Thus, $x \notin F_k$ for all k. However, $x \in \overline{B_{\delta_0}(x_0)} \subseteq E$, which is a contradiction. Thus, E contains no interior point. □